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Technical Notes

Modeled Lattice Boltzmann Equation and the Constant-Density Assumption

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Nomenclature

c	=	sound speed
D	=	flow dimension, 2 for two-dimensional flow and 3 for three-dimensional flow
f	=	particle distribution function
f^{eq}	=	equilibrium particle distribution function
h	=	half-width of the channel upstream of the expansion in a sudden expansion geometry
Kn	=	Knudsen number
L	=	characteristic length
M	=	Mach number
p	=	static pressure
Re	=	Reynolds number
t	=	time
$\boldsymbol{u}$	=	velocity vector
u	=	velocity component along x axis
v	=	velocity component along y axis
x, y	=	Cartesian coordinates
x_j	=	position vector
α	=	index for the lattice velocity
μ	=	fluid viscosity
$\boldsymbol{\xi}$	=	particle velocity vector
ξ_α	=	particle velocity along α lattice
ρ	=	fluid density
τ	=	relaxation time
τ_{ij}	=	viscous stress tensor
τ_o^*	=	reference relaxation time
ω	=	vorticity

Superscript

$*$	=	dimensional quantities

Subscript

∞	=	reference condition

Introduction

IT WAS pointed out by Chen et al. [1] that the lattice–gas fluid momentum equation [2] in the small velocity limit has two fundamental problems. The first is the non-Galilean invariance property of the equation because of the presence of a $g(\rho)$ coefficient in the nonlinear advection term, and the second is the explicit and unphysical velocity dependence of the pressure. Both problems could be remedied by adapting a lattice Boltzmann equation (LBE) method. Since then, most approaches to recover the Navier–Stokes (NS) equation have adopted the Bhatnagar–Gross–Krook (BGK)-type [3] modeled Boltzmann equation (BE) [4–7]. The BGK model proposes to represent the collision integral in the BE by the product of a relaxation frequency $1/\tau$ and a small deviation of f from f^{eq}. All approaches assume a Chapman–Enskog expansion for f in terms of Kn [8] and proceed to derive the NS equation from the second-order equation for f after showing that the solution of the zero-order equation for f is f^{eq}. For compressible flows in which M is not small, the NS equation thus obtained is accurate to $\mathcal{O}(M^2)$, because this is the order of the nonlinear deviation term [7]. The method also yields expressions for the sound speed and the shear viscosity.

For incompressible flows, the claim was made that it is not possible to maintain a constant ρ in the lattice Boltzmann model; therefore, the LBE always simulates the compressible NS equation [5]. This is because the spatial variation of $\rho(\delta\rho)$ is not zero in any LBE simulations. To simulate incompressible flows correctly, He and Luo [5] suggested that M and $\delta\rho$ were of $\mathcal{O}(Kn)$ and $\mathcal{O}(Kn^2)$, respectively. Two attempts have been made to achieve this objective. The first approach [4] is to recast the NS equation in terms of the momentum density ($\boldsymbol{j} = \rho\boldsymbol{u}$). The second approach [5] argues that the constant-ρ limit is equivalent to the $M \ll 1$ assumption; therefore, it is only necessary to recover the NS equation (written in terms of $\boldsymbol{u}$) accurate to $\mathcal{O}(M^2)$ in the mass conservation equation and to $\mathcal{O}(M^3)$ in the momentum equation. In so doing, Mach number is implicitly assumed to be related to Knudsen number; the physical reason for this assumption is not clear. The first approach is not satisfactory because its validity for unsteady flow is not known. On the other hand, the assumption of the second approach immediately poses a problem when the dimensionless compressible NS equation is used to recover its incompressible counterpart. In this set of equations, Mach number appears with Reynolds number and Prandtl number as coefficients in the viscous and heat conduction term of the momentum and energy equation. Taking the limit of $M \to 0$ is tantamount to reducing the equations to their inviscid counterparts; thus, it can be seen that the constant-ρ limit is not equivalent to the $M \ll 1$ assumption [9].

It is quite often argued that in the $M \to 0$ limit, the isothermal gas law $p = c^2\rho$ is valid for both gas and liquid because only the pressure gradient is of importance in incompressible flows. Further, since c is usually very large, $\delta\rho$ is essentially negligible for any practical variation of δp. The second approach [5] avoids the pitfall of relating p directly to ρ by postulating a relation between p and f_α. Thus, the formulation was carried out in terms of p_α and expanding p_α^{eq} in terms of $\boldsymbol{\xi}_\alpha \cdot \boldsymbol{u}$. This approach yields good result using a two-dimensional (2-D), nine-velocity-lattice (D2Q9) model to simulate 2-D flows. However, the physical justification for expressing p (δp is the driving potential in an incompressible flow) as a distribution function of $\boldsymbol{\xi}_\alpha \cdot \boldsymbol{u}$ is not clear.

This Note reports on an approach to recover the incompressible NS equation based on a constant-ρ assumption. The recovered NS equation is identical to the continuum equation, and the only limitation is $Kn \ll 1$. The vehicle to achieve this is to seek a f_α^{eq} that is not an expansion in terms of $\boldsymbol{\xi}_\alpha \cdot \boldsymbol{u}$, but a polynomial of ξ_α. The constant-ρ assumption can then be used to evaluate the pressure and there is no need to postulate a p distribution function or to assume that p is related to ρ (or $\boldsymbol{u}$). With an eye on the ease with which boundary conditions can be set for the LBE, a splitting method similar to Toro's [10] proposal is used to solve the LBE; the simulations are validated against 2-D unsteady flow problems.

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Recovery of the Incompressible Navier–Stokes Equation

The starting point of the present formulation is the unsteady incompressible isothermal NS equation, which in dimensionless conservation form can be written as

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{1}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + \frac{1}{Re}\Im_{ij}\right] \tag{2}$$

Equations (1) and (2) are normalized using L^*, c^*_∞, ρ^*_∞, and L^*/c^*_∞ as characteristic length, velocity, density, and time, respectively. Therefore, the dimensionless variables are

$$x_j = x^*_j/L^*, \qquad t = t^*/(L^*/c^*_\infty), \qquad u_j = u^*_j/c^*_\infty$$
$$\rho = \rho^*/\rho^*_\infty, \qquad p = p^*/\rho^*(c^*_\infty)^2, \qquad Re = \rho^* c^*_\infty L^*/\mu^*$$
$$S^*_{ij} = (1/2)(\partial u^*_i/\partial x_j + \partial u^*_j/\partial x_i)$$
$$\Im_{ij}/Re = [2(\mu^*/\rho^*)(S^*_{ij} - (1/3)S^*_{kk}\delta_{ij})]/(c^*_\infty)^2$$

For purpose of generality, ρ^*_∞ is assumed to be different from ρ^*; therefore, setting $\rho = 1$ yields the familiar incompressible NS equation for a gas or a liquid. Note, there is no stipulation that p^* has to be related to ρ^*.

The objective is to recover Eqs. (1) and (2) from a BGK-type modeled BE. This modeled BE can be written in dimensionless form as [1]

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi}\cdot\nabla_x f = -\frac{1}{\tau Kn}(f - f^{\text{eq}}) \tag{3}$$

where $\tau = \tau^*/\tau^*_o$, $\boldsymbol{\xi} = \boldsymbol{\xi}^*/c^*_\infty$, and

$$(f, f^{\text{eq}}) = (f^*, (f^*)^{\text{eq}})/(\rho^*_\infty/(c^*_\infty)^D)$$

have been substituted. The Chapman–Enskog expansion [8] is then applied to f, such that

$$f = f^{(0)} + Knf^{(1)} + Kn^2 f^{(2)} + \mathcal{O}(Kn^3) \tag{4}$$

and the following equations are obtained:

$$f^{(0)} = f^{\text{eq}}, \quad \text{to } \mathcal{O}(Kn^0) \tag{5a}$$

$$\frac{\partial f^{(0)}}{\partial t} + \boldsymbol{\xi}\cdot\nabla_x f^{(0)} = -\frac{f^{(1)}}{\tau} \quad \text{to } \mathcal{O}(Kn^1) \tag{5b}$$

Similar to the approach adopted for the recovery of the compressible Euler equation [11,12], the incompressible NS Eqs. (1) and (2) can be recovered identically by multiplying Eq. (5b) by $[1, \quad \boldsymbol{\xi}]^T$, then integrating over the whole particle velocity space and making use of the following constraints,

$$\int f^{\text{eq}}\mathrm{d}\boldsymbol{\xi} = \rho \tag{6}$$

$$\int f^{\text{eq}}\xi_i\mathrm{d}\boldsymbol{\xi} = \rho u_i \tag{7}$$

$$\int \xi_i\xi_j f^{\text{eq}}\mathrm{d}\boldsymbol{\xi} = p\delta_{ij} + \rho u_i u_j - \tau_{ij} \tag{8}$$

with the substitution $\tau_{ij} = \Im_{ij}/Re$ and the following assumptions:

$$\int f^{(n)}\mathrm{d}\boldsymbol{\xi} = 0 \tag{9a}$$

$$\int f^{(n)}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\xi} = 0 \quad \text{for all } n \geq 1 \tag{9b}$$

Equation (6) is simply a constraint on f^{eq} that guarantees mass conservation and, as will be seen later, yields a correct calculation of the driving potential. In addition, Eq. (8) is used as a constraint to ensure that the incompressible NS equation is recovered correctly. There is no need to invoke $p^* = \rho^*(c^*_\infty)^2$, which is only correct for isothermal perfect gas [1]. Consequently, Eq. (6) can be used to determine the pressure field through a pressure-correction method similar to the proposal of Patankar and Spalding [13]. It is in this sense that the constant-ρ assumption is imposed and is the basic difference between the present approach and those proposed previously [4,5].

It is clear that there is no need to use the $f^{(2)}$ equation to recover Eqs. (1) and (2). The equations thus recovered are correct to order Kn, which is simply a consequence of the continuum assumption. Otherwise, there are no other restrictions on the recovered NS equation. Just as in the case of solving the continuum NS equation, the reference Reynolds number has to be specified. This ensures that μ is correctly specified in the whole flowfield. The next step is to seek a f^{eq} that satisfies Eqs. (6–9). This will be carried out only in the lattice approach to solve Eq. (3). As later derivation will bear out, under this formulation, the coefficients of f^{eq} for incompressible isothermal flow are different from those deduced for compressible flow [14]; that is, there is no one general set of coefficients valid for incompressible and compressible flows.

Determination of Lattice f^{eq}

To solve Eq. (3) numerically, a lattice approach is adopted. Therefore, it is necessary to write down the lattice counterpart of Eq. (3). This is given by

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha\cdot\nabla_x f_\alpha = -\frac{1}{\tau Kn}(f_\alpha - f^{\text{eq}}_\alpha) \tag{10}$$

The corresponding lattice form for Eq. (4) is

$$f_\alpha = f^{\text{eq}}_\alpha + Knf^{(1)}_\alpha + Kn^2 f^{(2)}_\alpha + \mathcal{O}(Kn^3) \tag{11}$$

In line with [11,12], a second-order polynomial in $\boldsymbol{\xi}_\alpha$ is assumed for the lattice f^{eq}: that is,

$$f^{\text{eq}}_\alpha = A_\alpha + \xi_{\alpha x}Ax_\alpha + \xi_{\alpha y}Ay_\alpha + \xi^2_{\alpha x}Bxx_\alpha + \xi^2_{\alpha y}Byy_\alpha + \xi_{\alpha x}\xi_{\alpha y}Bxy_\alpha \tag{12}$$

where the coefficients A_α, Ax_α, Ay_α, etc., are not known and need to be determined. The constraints used to evaluate these coefficients are given by Eqs. (6–9). Since these coefficients are expected to depend on the velocity lattice model assumed, their values for a D2Q9 model are determined by assuming the lattice distribution and its magnitude (where σ is a parameter to be specified) to be given by

$$\boldsymbol{\xi}_0 = 0, \qquad \alpha = 0 \tag{13a}$$

$$\boldsymbol{\xi}_\alpha = \sigma\{\cos[\pi(\alpha-1)/4], \sin[\pi(\alpha-1)/4]\}, \qquad \alpha = 1,3,5,7 \tag{13b}$$

$$\boldsymbol{\xi}_\alpha = \sqrt{2}\sigma\{\cos[\pi(\alpha-1)/4], \sin[\pi(\alpha-1)/4]\}, \qquad \alpha = 2,4,6,8 \tag{13c}$$

Before evaluating the coefficients, it is necessary to show how the 2-D NS equation can be recovered from Eq. (12). First, it is assumed that for each α, f is given by Eq. (11) and the constraints, Eqs. (6–9), are used to evaluate the coefficients in Eq. (12). For 2-D flows, the results are given by

$$\sum_{\alpha=0}^{N} f^{\text{eq}}_\alpha = \rho = \text{constant} \tag{14a}$$

$$\sum_{\alpha=0}^{N} f^{\text{eq}}_\alpha \xi_{\alpha x} = \rho u \tag{14b}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{\text{eq}} \xi_{\alpha y} = \rho v \tag{14c}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{\text{eq}} \xi_{\alpha x}^2 = \rho u^2 + p - \tau_{xx} \tag{14d}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{\text{eq}} \xi_{\alpha y}^2 = \rho v^2 + p - \tau_{yy} \tag{14e}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{\text{eq}} \xi_{\alpha x} \xi_{\alpha y} = \rho u v - \tau_{xy} \tag{14f}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{(n)} = 0 \tag{14g}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{(n)} \xi_{\alpha x} = 0 \tag{14h}$$

$$\sum_{\alpha=0}^{N} f_\alpha^{(n)} \xi_{\alpha y} = 0 \quad \text{for } n \geq 1 \tag{14i}$$

where summation is carried out over $0 \leq N \leq 8$ for a D2Q9 model. Multiplying Eq. (10) with respect to $[1, \quad \boldsymbol{\xi}_\alpha]^T$, taking summation over α, and making use of Eq. (14a–14i), the final macro transport equations for a 2-D flow in Cartesian coordinates are obtained:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + \mathcal{O}(Kn) \tag{15a}$$

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \mathcal{O}(Kn) \tag{15b}$$

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \mathcal{O}(Kn) \tag{15c}$$

For a D2Q9 model, there are eight values to each coefficient. If the coefficients having the same energy shell of the lattice velocities are assumed identical, the number of unknowns resulting from the coefficients A_α, Ax_α, Bxx_α, etc., are 13 in a D2Q9 lattice model. Since the number of constraints available for the determination of these coefficients is six, there is certain flexibility and assumptions can be made to facilitate solution of the equations. As a first attempt, seven coefficients out of the 13 are assumed to be 0. The results are

$$A_0 = \rho - \frac{2p}{\sigma^2} - \frac{\rho|\boldsymbol{u}|^2}{\sigma^2} + \frac{\tau_{xx} + \tau_{yy}}{\sigma^2}, \qquad A_1 = A_2 = 0 \tag{16a}$$

$$Ax_1 = \frac{\rho u}{2\sigma^2}, \qquad Ax_2 = 0 \tag{16b}$$

$$Ay_1 = \frac{\rho v}{2\sigma^2}, \qquad Ay_2 = 0 \tag{16c}$$

$$Bxx_1 = \frac{1}{2\sigma^4}(p + \rho u^2 - \tau_{xx}), \qquad Bxx_2 = 0 \tag{16d}$$

$$Byy_1 = \frac{1}{2\sigma^4}(p + \rho v^2 - \tau_{yy}), \qquad Byy_2 = 0 \tag{16e}$$

$$Bxy_2 = \frac{1}{4\sigma^4}(\rho uv - \tau_{xy}), \qquad Bxy_1 = 0 \tag{16f}$$

The six coefficients do not contain any arbitrary constants except σ, which is as yet undefined. However, its value, which affects numerical stability, can be estimated from Eq. (13b), (14d), and (14e); the result is given by $\sigma^2 = |\boldsymbol{\xi}_\alpha|^2$. It should be noted that this set of coefficients is not unique. Other assumptions could be made for the seven zero coefficients. However, the ultimate test is the correctness of the simulation results compared to known solutions. This set of coefficients led to simulation results that are in agreement with flow cases attempted in this Note and in [15].

Numerical Method and Simulation Results

Equation (10) can be solved using any numerical method including those adopted for the conventional lattice Boltzmann method (LBM) [16] and the previously proposed finite difference lattice Boltzmann method (FDLBM) [11,12]. However, attention is paid to the ease with which the boundary condition for f can be set. For some flow cases, a corresponding boundary f_α might be difficult to define if conventional LBM is used. The choice of the numerical method, therefore, hinges on whether the boundary condition for Eq. (10) can be set as conveniently as conventional finite difference schemes used in DNS. A brief description of the FDLBM and the boundary conditions is given next; this is followed by a discussion of two validation cases.

Numerical Scheme

One such scheme is the splitting method of Toro [10], which splits the inhomogeneous Eq. (10) into its homogeneous counterpart and a diffusion equation. This same scheme has been incorporated into previously proposed FDLBM [11,12,15], in which steady incompressible microchannel flows are simulated. The present approach will adopt this modified FDLBM with the proposed modeled BE. Details of the numerical procedure can be found in [15]; therefore, only the salient steps are described next.

1) Initial f_α^{eq}, which are calculated by the given initial macroscopic quantities, are used as initial values to start the calculation.

2) With f_α at time t known, an intermediate value f_α^I is calculated from the equation

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla_{\mathbf{x}} f_\alpha = 0 \tag{17}$$

which can be solved by any numerical scheme.

3) Using this f_α^I, the corresponding intermediate macroscopic quantities (u_I, v_I, p_I) for all interior grid points are calculated as

$$u_I \equiv \frac{1}{\rho} \sum_{\alpha=0}^{8} f_\alpha^I \xi_{\alpha x} \tag{18a}$$

$$v_I \equiv \frac{1}{\rho} \sum_{\alpha=0}^{8} f_\alpha^I \xi_{\alpha y} \tag{18b}$$

$$p_I \equiv \sum_{\alpha=0}^{8} f_\alpha^I \frac{1}{2}(\xi_{\alpha x}^2 + \xi_{\alpha y}^2) - \frac{1}{2}\rho|\boldsymbol{u}_I|^2 + \frac{\tau_{xxI} + \tau_{yyI}}{2} \tag{18c}$$

4) The boundary conditions for the macroscopic level are then set as in any finite difference methods.

5) Using the macroscopic quantities thus determined, a corresponding $f_\alpha^{I,\text{eq}}$ is obtained.

6) Using f_α^I as the initial condition, the diffusion equation

$$\frac{\partial f_\alpha}{\partial t} = -\frac{1}{\tau Kn}(f_\alpha - f_\alpha^{\text{eq}}) \tag{19}$$

is solved. The scheme makes use of the advantage of an arbitrary relaxation time. By setting $\tau = 1$ and $Kn = \Delta t$, it can be shown that f_α at time $(t + \Delta t)$ is exactly the same as $f_\alpha^{I,\text{eq}}$ (see [15] for details); hence,

$$(u, v, p)|_{t+\Delta t} = (u_I, v_I, p_I)$$

7) Time marching proceeds by repeating procedures 2 to 6.

Unlike the conventional LBM, the density ρ is not determined by f_α, but rather it is required to be a given constant. Thus, if there is a deviation between the given ρ and the value of Σf_α, this difference is called the artificial density. Because of the definition of the macroscopic quantities in step 3, Eqs. (14g–14i) might not be satisfied; rather, the following is obtained:

$$\sum_{\alpha}(f_\alpha - f_\alpha^{eq})\begin{pmatrix}1\\ \xi_{\alpha x}\\ \xi_{\alpha y}\\ \frac{\xi_{\alpha x}^2+\xi_{\alpha y}^2}{2}\end{pmatrix} = \begin{pmatrix}\sum_\alpha f_\alpha - \rho\\ 0\\ 0\\ 0\end{pmatrix} \tag{20}$$

As a result, the use of Eq. (9b) implies

$$\frac{\partial \sum_\alpha f_\alpha}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = -\frac{\sum_\alpha f_\alpha - \rho}{Kn} \tag{21a}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2 + p - \tau_{xx}}{\partial x} + \frac{\partial \rho uv - \tau_{xy}}{\partial y} = \mathcal{O}(Kn) \tag{21b}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv - \tau_{xy}}{\partial x} + \frac{\partial \rho v^2 + p - \tau_{yy}}{\partial y} = \mathcal{O}(Kn) \tag{21c}$$

$$\frac{\partial}{\partial t}\left[p + \frac{1}{2}\rho|\mathbf{u}|^2 - \frac{\tau_{xx} + \tau_{yy}}{2}\right] + \frac{\sigma^2}{2}\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right) = \mathcal{O}(Kn) \tag{21d}$$

The last equation is due to the fact that

$$\sum_{\alpha=0}^{8} f_\alpha^{eq}\frac{\xi_{\alpha x}}{2}(\xi_{\alpha x}^2 + \xi_{\alpha y}^2) = \frac{\sigma^2 \rho u}{2} \tag{22a}$$

$$\sum_{\alpha=0}^{8} f_\alpha^{eq}\frac{\xi_{\alpha y}}{2}(\xi_{\alpha x}^2 + \xi_{\alpha y}^2) = \frac{\sigma^2 \rho v}{2} \tag{22b}$$

which is a consequence of the choice of parameters in Eqs. (16a–16f).

Equations (21b) and (21c) are satisfied with an error of $\mathcal{O}(Kn)$. However, it should be noted that Eq. (21a) is not the continuity equation. Even if the difference between ρ and Σf_α is very small [say, of $\mathcal{O}(Kn)$], the right-hand side is still significant. Treating the evolution as an iteration process and solving the equations until a steady state has been reached, the temporal derivative in Eq. (21d) vanishes. The fluctuation of this term is used as a convergence criterion for the iteration. As steady state is reached, the artificial density approaches the actual density and mass conservation is satisfied. For a large enough σ, which depends on numerical stability, iterations could be limited to a few, or perhaps no iteration is required for the satisfaction of Eq. (21a). Thus, the scheme becomes time-accurate (see the Stokes second problem, next). Together, Eqs. (21a) and (21d) function similarly to the pressure-correction method of [13]. Two test cases are used to validate this approach: they are the Stokes second problem and a 2-D sudden expansion flow.

Stokes Second Problem

In this simulation, the x axis coincides with an infinitely long flat plate above which is a viscous fluid. The characteristic length, velocity, and time are given by $L^* = 1$, $c_\infty^* = U^*$, and $L^*/c_\infty^* = 1/U^*$, respectively. The boundary at $y = 0$ is oscillating in time such that

$$u(y=0,t) = U\cos nt \tag{23}$$

An exact solution for this problem is given by

$$u(y,t) = U\exp\left(-\sqrt{\frac{nRe}{2}}y\right)\cos\left(nt - \sqrt{\frac{nRe}{2}}y\right) \tag{24}$$

The domain is bounded by $0 \le x, y \le 1$ in the numerical calculation. In step 2, Eq. (17) is discretized in much the same manner as the conventional LBM [16]. The numerical and physical parameters are given by $\Delta x = 0.01$, $\Delta t = 0.00001$, $\sigma = \Delta x/\Delta t = 1000$, $Re = 20$, $U = 1$, and $n = 4\pi$. Comparisons of the calculated results with the analytical solution at various time intervals are shown in Fig. 1. It can be seen that the FDLBM simulations are in excellent agreement at every t, compared with the analytical results given by Eq. (24).

Two-Dimensional Sudden Expansion Flow

The flow geometry of a symmetric 2-D sudden expansion is shown in Fig. 2; the characteristic length, velocity, and time are given by $L^* = 1$, $c_\infty^* = U_{ave}^*$ (average velocity of the inflow), and $L^*/c_\infty^* = 1/U_{ave}^*$, respectively. The entry flow (at $x = 0$) is taken to be parabolic and its profile is given by

$$u|_{x=0} = \begin{cases}\frac{3}{2h}\left[1-\left(\frac{y}{h}\right)^2\right], & 0 \le y \le h\\ 0, & h \le y \le 1\end{cases} \tag{25}$$

Parabolic condition is specified as the downstream boundary condition; that is, all variables except p are independent of x. Hence, a long channel is required for the flow to become fully developed after the expansion. The width of the downstream channel is taken to be two, thus giving an expansion ratio of $1/h$. The entire channel is symmetric about the x axis ($y = 0$). A computational domain of $0 \le x \le 20$, $0 \le y \le 1$ is used. The Lax and Wendroff [17] scheme is used to solve Eq. (17) with numerical and physical parameters specified by $\Delta x = 0.05$, $\Delta t = 0.00001$, $h = 0.5$, and $Re = 46.6$. For the present investigation, in search of better accuracy, σ is allowed to vary for each iteration step according to

$$\sigma = 2.5\sqrt{\max_{(x,y)}\left\{u^2 + v^2 + \frac{2p - \tau_{xx} - \tau_{yy}}{\rho}\right\}} \tag{26}$$

rather than adopting a constant σ as in [15]. This test case has also been reported in [15], but due to the special treatment of σ and a longer channel employed, the result reported here is much improved. The reattachment length l, the location of the eddy-center l_e, and the minimum vorticity ω_{min} at the upper wall of the channel are tabulated for comparison with other known data in Table 1. It should be noted that the accuracy of the FDLBM simulations is greatly affected by the grid size; a change of the grid size from $\Delta x = 0.001$ to 0.01 leads to a completely incorrect capture of the shock in a 1-D Riemann problem

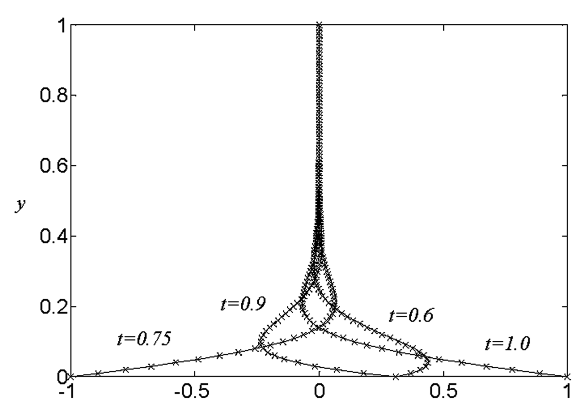

Fig. 1 Distribution of the *u* velocity along the *y* axis at different times: solid line represents exact solution and *x* represents FDLBM simulation results.

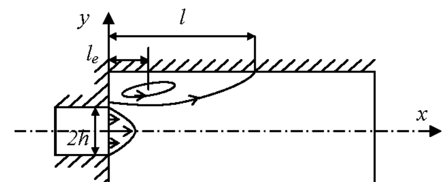

Fig. 2 Schematic diagram of the flow in a 2-D sudden expansion.

Table 1 Comparison of the calculated l, l_e, and $\omega_{\min}$ with known data

	l/Re	l_e/Re	$\omega_{\min}$
Current FDLBM result	0.0689	0.016	−2.52
Agarwal [18]	0.068	0.016	−2.26
Kumar and Yajnik ($N = 3$) [19]	0.064	0.014	−2.19
Hung and Macagno [20]	0.066	0.013	−2.82

[12]. The present simulation is carried out using $\Delta x = 0.05$, and since there is no complete agreement among other known data [18–20] (Table 1), the comparison with these results can be considered quite well. No doubt, reducing Δx could further improve the predictions of l, l_e, and $\omega_{\min}$. Therefore, the present results show that the constant-ρ assumption leads to a correct recovery of the incompressible NS equation.

Conclusions

It is shown that the BGK-type modeled BE can be used to recover the correct incompressible NS equation by just invoking constant density. The only restriction on the recovered NS equation is $Kn \ll 1$, equivalent to the continuum assumption. This is accomplished by adopting an expression for f^{eq} that is a second-order polynomial in $\boldsymbol{\xi}_\alpha$; it is noted that the expression assumed for f^{eq} is not an expansion in $\boldsymbol{\xi}_\alpha \cdot \boldsymbol{u}$. The 3-D incompressible NS equation can be similarly derived. The BGK-type modeled BE is solved using a finite difference lattice Boltzmann method. The simulations of a 2-D sudden expansion flow and Stokes second problem are compared with known numerical and analytical solutions, and agreement with both solutions is obtained.

It is worthy to note that the coefficients thus determined for Eq. (10) are different from those given in [14] for compressible flows. They are also different from those given in [11,12] for inviscid compressible flows, even after $1/Re = 0$ has been set in Eqs. (16a–16f) to reduce the incompressible NS equation to its Euler counterpart. It could be further deduced that the presence of heat transfer in the flow would also alter the coefficients. Therefore, these results show that a general set of coefficients equally valid for incompressible (isothermal and nonisothermal) and compressible flows is not available.

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